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Images and Inverse images in the Category of Fuzzy groups

P.Vijayalakshmi, Dr. P. Alphonse Rajendran

Abstract— Ever since fuzzy sets were introduced by Lotfi Zadeh in the year 1965 [1], many algebraic structures were introduced by many authors . One such structure is fuzzy groups introduced in [2] and [3]. In [4] the authors introduced a novel definition of fuzzy group homomorphism between any two fuzzy groups and gave element wise characterization of some special morphisms in the category of fuzzy groups

Index Terms— Epimorphism, Epimorphic images, Fuzzy sub group, Fuzzy group homomorphism, Images, Inverse Images, , Monomorphism, Strong Monomorphism

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1 Introduction

In this article we prove that the category of fuzzy groups has epimorphic images and inverse images. We begin with the following definitions.

In [3], Azriel Rosenfeld has defined ***a fuzzy subgroup*** μ on a group S where $\mu : S \to [0,1]$ is a function as one which satisfies $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$.
Equivalently by proposition 5.6 in [3]
(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(x^{-1}) = \mu(x)$

We take this as the definition of a fuzzy group. However in our notation and terminology for fuzzy sets a ***fuzzy group*** in this article will be a pair
$(X, \mu) = \{(x, \mu(x)) \; / \; x \in X, \mu : X \to [0,1]$ is a function }, where
(i) X is a group and
(ii) $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ for all x, y ∈ X

Let (X, μ) and (Y, η) be fuzzy groups. Then a ***fuzzy group homomorphism*** from (X, μ) into (Y, η) is a pair (f, α) where $f : X \to Y$ is a group homomorphism (in the crisp sense) and $\alpha : \mu(X) \to \eta(Y)$ is a function such that $\alpha\mu = \eta f$. Equivalently $(f, \alpha) : (X, \mu) \to (Y, \eta)$ is a fuzzy group homomorphism (or fuzzy morphism) if $f : X \to Y$ is a homomorphism (crisp) of groups and the following diagram commutes.

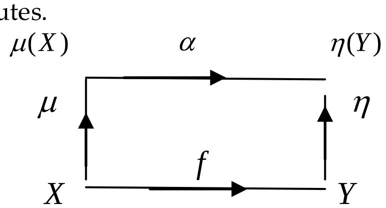

fig. 1

2 Definitions

Definition 2.1

A fuzzy morphism $(f, \alpha) : (X, \mu) \to (Y, \eta)$ is called a **monomorphism** in $\mathcal{F}$ if for all pairs of fuzzymorphisms (g, β) and $(h, \delta) : (Z, \theta) \to (X, \mu), (f, \alpha)(g, \beta) = (f, \alpha)(h, \delta)$ implies that $(g, \beta) = (h, \delta)$ {(i.e). (f, α) is left cancellable in $\mathcal{F}$}.

A monomorphism is called a **strong monomorphism** if (f, α) is injective.

Definition 2.2

Let $(f, \alpha) : (X, \mu) \to (Y, \eta)$ be a given fuzzy group homomorphism and $(u, \delta) : (I, \xi) \to (Y, \eta)$ be a fuzzy subgroup of (Y, η). Then (I, ξ) is called an ***image*** of (f, α) if

(i) $(f, \alpha) = (u, s)(f_1, \alpha_1)$ for some fuzzy group homomorphism $(f_1, \alpha_1) : (X, \mu) \to (I, \xi)$

(ii) If $(v, t) : (J, \theta) \to (Y, \eta)$ is any fuzzy subgroup of (Y, η) such that $(f, \alpha) = (v, t)(g, \beta)$ for some fuzzy group homomorphism $(g, \beta) : (X, \mu) \to (J, \theta)$ then there exists a fuzzy group homomorphism $(h, \gamma) : (I, \xi) \to (J, \theta)$ such that $(u, s) = (v, t)(h, \gamma)$.

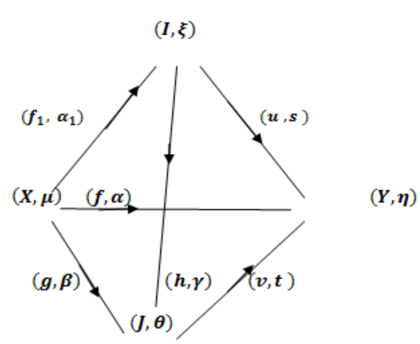

fig. 2

- P.Vijayalakshmi, Research Scholar, Periyar Maniammai University, Periyar Nagar, Vallam, Thanjavur, Tamil Nadu, India, Mobile no:9790035790.
- Dr. P. Alphonse Rajendran, Former Prof.(Dean SHSM),Department of Mathematics, Periyar Maniammai University,Periyar Nagar, Vallam, Thanjavur, Tamil Nadu, India. Mobile no:9790035789

Definition 2.3 : Let $(f,\alpha) : (X , \mu) \to (Y , \eta)$ be a given fuzzy group homomorphism and $(u,\delta) : (Z , \theta) \to (Y , \eta)$ be a fuzzy sub group. An object (P,ϵ) in $\mathcal{F}$ is called the inverse image of (Z,θ) by (f,α) if there exists morphisms $(p_1,\beta_1) : (P,\epsilon) \to (Z,\theta)$ and $(p_2,\beta_2) : (P,\epsilon) \to (X,\mu)$ such that

(i). $(u,\delta)(p_1,\beta_1) : (f,\alpha)(p_2,\beta_2)$ and

(ii). if there exists morphisms $(q_1,\delta_1) : (Q,\xi) \to (Z,\theta)$ and $(q_2,\delta_2) : (Q,\xi) \to (X,\mu)$ such that $(u,\delta)(q_1,\delta_1)=(f,\alpha)(q_2,\delta_2)$ then there exists a unique fuzzy group homomorphism $(h,\gamma): (Q,\xi) \to (p,\varepsilon)$ such that $(p_1,\beta_1)(h,\gamma) = (q_1,\delta_1)$ and

$(p_2,\beta_2)(h,\gamma) = (q_2,\delta_2)$.

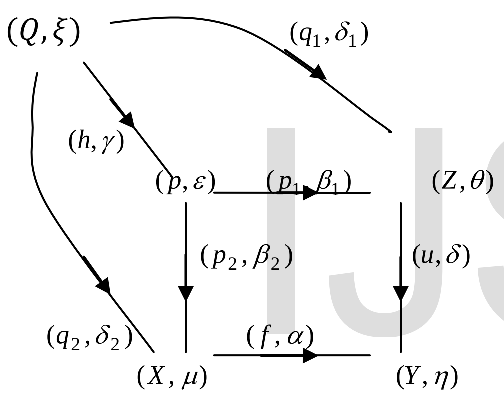

fig. 3

Remark 2.4:

(a) Since (I,ξ) and (J,θ) are fuzzy subgroups of (Y,η), (u,s) and (v,t) are strong monomorphisms.

(b) From (ii) we have $vh = u$ is injective and $t\gamma = s$ is also injective , both h and γ are injective. Thus (h,γ) is a strong monomorphism.[definition of strong monomorphism]

(c) Since (v,t) is a (strong) monomorphism. (h,γ) with the above property in (ii) is unique.

A category $\mathcal{A}$ is said to have images if every morphism in that category has an image . Moreover, if in the factorization $(f,\alpha)=(u,s)(f_1,\alpha_1)$, the morphism (f_1,α_1) is always an epimorphism, then the category $\mathcal{A}$ is said to have epimorphic images. We now prove

3 THEOREMS

Theorem 3.1: The category of fuzzy groups has epimorphic images.

Proof. We prove the theorem via two lemmas.

Lemma 3.2: The category of fuzzy groups has images.

Proof. Let $(f,\alpha) : (X, \mu) \to (Y,\eta)$ be any given fuzzy group homomorphism.

Let $f(X) = \{{}^{f(x)}/_{x \in X}\}$.

Define $f_1 : X \to f(X)$, as $f_1(x) = f(x)$ for all $x \in X$.

$\eta_1 : f(X) \to [0,1]$ as $\eta_1 f(x) = \eta f(x)$ and

$\alpha_1 : \mu(X) \to \eta_1 f(X)$ as $\alpha_1\mu(x) = \eta_1 f_1(x) = \eta f(x)$

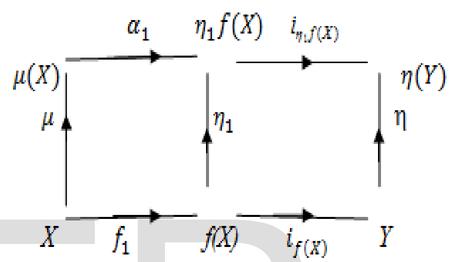

fig. 4

Let $i_{f(X)}: f(X) \to Y$ and $i_{\eta_1 f(X)} : \eta_1 f(X) \to \eta(Y)$ be the respective inclusion maps. Then $(f(X),\eta_1)$ is a fuzzy subgroup of (Y,η). In fact $(i_{f(X)}, i_{\eta_1 f(X)})$ is a strong monomorphism.

Claim. $(i_{f(X)}, i_{\eta_1 f(X)}) : (f(X),\eta_1) \to (Y,\eta)$ is an image of (f,α).

Now from the definitions, it follows that for all $x \in X$, $i_{f(X)}\ f_1(x) = i_{f(X)}\ f(x) = f(x)$

so that $i_{f(X)} \circ f_1 = f$. Similarly $i_{\eta_1 f(X)} \circ \alpha_1 = \alpha$ so that $(i_{f(X)}, i_{\eta_1 f(X)})\ (f_1,\alpha_1) = (f,\alpha)$.

Thus condition (i) of definition 1.2 is satisfied.

Suppose there exists a morphism $(g,\beta):(X,\mu) \to (J,\theta)$ and a strong monomorphism $(\vartheta,t):(J,\theta) \to (Y,\eta)$ such that $(\vartheta,t)(g,\beta) = (f,\alpha)$ (1)

Define $h: f(X) \to J$ by $h\,f(x) = g(x)$ (2)

Then h is well defined.

For $f(x_1) = f(x_2)$

$\Rightarrow \vartheta g(x_1) = \vartheta g(x_2)$ [from (1)]

$\Rightarrow g(x_1) = g(x_2)$[since ϑ is injective].

Moreover Since $g: X \to J$ is a homomorphism of groups, h is also a homomorphism of groups.

Also for all $x \in X$, $\vartheta hf(x) = \vartheta g(x)$ (by (2))

$= f(x)$ (by (1))

$$\Longrightarrow vh = i_{f(X)} \quad (3)$$

Again we define $\gamma: \eta_1(f(X)) \to \theta(J)$ as follows

Given x ∈ X, $\eta_1 f(x) = \eta f(x)$

$= \eta v g(x)$ (by (1))

$= t\theta g(x)$ [since $\eta\vartheta = t\theta$]

Hence for each $x \in X$, there is a unique θg(x) ∈ $\theta(J)$[since t is injective] such that $\eta_1 f(x) = t\theta g(x)$ (4)

So define

$$\gamma : \eta_1 f(X) \to \theta(J) \text{ by } \gamma\eta_1 f(x) = \theta g(x) \quad (5)$$

[γ is well defined since $\eta f(x_1) = \eta f(x_2)$

$\Rightarrow \eta\vartheta g(x_1) = \eta\vartheta g(x_2)$

$\Rightarrow t\theta g(x_1) = t\theta g(x_2)$

$\Rightarrow \theta g(x_1) = \theta g(x_2)$]

Moreover for all x ∈ $X, \gamma\eta_1 f(x) = \gamma\eta f(x) = \theta g(x)$ [definition of γ]

$= \theta hf(x)$ [since $g = hf$]

and so $\gamma\eta_1 = \theta h$.

Thus $(h,\gamma):(f(x),\eta_1) \to (J,\theta)$ is a fuzzy group homomorphism.

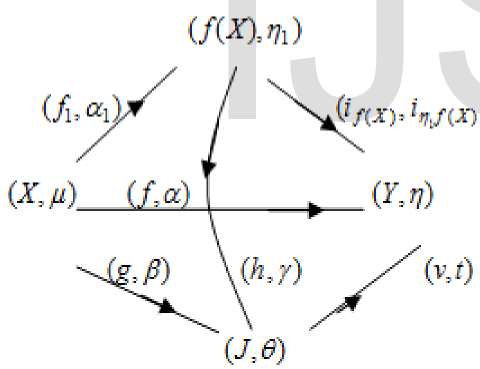

fig. 5

Finally for all $x \in X, t\gamma(\eta_1 f(x)) = t\theta g(x)$

[by (5)]

$= \eta\vartheta g(x)$ [since $t\theta = \eta v$]

$= \eta f(x)$ [since $vg = f$]

$= \eta_1 f(x)$

which implies that

$$t\gamma = (i_{f(x)}, i_{mf(x)}) \quad (6)$$

from (3) and (6) we have

$$(v,t)(h,\gamma) = (i_{f(x)}, i_{mf(x)})$$

Thus condition (ii) of definition 1.2 is also satisfied.

Thus the category of fuzzy groups say $\mathcal{F}$ has images.

Lemma 3.3: Let $(f,\alpha)(X,\mu) \to (Y,\eta)$ be a fuzzy group homomorphism and let

$$(X,\mu) \xrightarrow{(f_1,\alpha_1)} (I,\xi) \xrightarrow{(u,s)} (Y,\eta)$$

be a factorization of (f,α) through its image $(u,s):(I,\xi) \to (Y,\eta)$. Then (f_1,α_1) is an epimorphism.

proof. Let $(g_1,\beta_1),(g_2,\beta_2):(I,\xi) \to (C,\tau)$ be fuzzy group homomorphism such that

$$(g_1,\beta_1)(f_1,\alpha_1) = (g_2,\beta_2)(f_1,\alpha_1) \quad (7)$$

Let $(h,\gamma):(E,\theta) \to (I,\xi)$ be the equalizer for (g_1,β_1) and (g_2,β_2) [This exists by [4]]

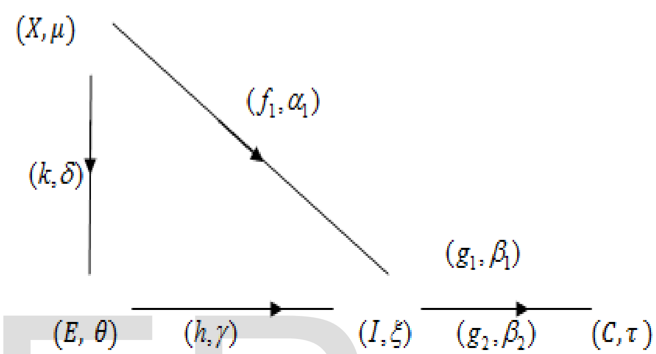

fig. 6

Then $(g_1,\beta_1)(h,\gamma) = (g_2,\beta_2)(h,\gamma)$ (8)

Now from (1) and the definition of an equalizer there exists a unique fuzzy group homomorphism

$(k,\delta):(X,\mu) \to (E,\theta)$ such that

$$(h,\gamma)(k,\delta) = (f_1,\alpha_1) \quad (9)$$

Hence $(u,s)(h,\gamma)(k,\delta) = (u,s)(f_1,\alpha_1) = (f,\alpha)$ (10)

Thus (f,α) factors through (E,θ).

Therefore by the definition of an image , there exists a unique fuzzy group homomorphism $(p,\omega):(I,\xi) \to (E,\theta)$ such that $(u,s)(h,\gamma)(p,\omega) = (k,s)$

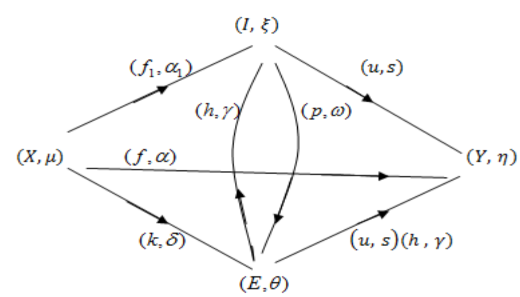

fig. 7

This implies that $(h,\gamma)(p,\omega)$ = identity on (I,ξ) (11)

Now from (2) $(g_1,\beta_1)(h,\gamma) = (g_2,\beta_2)(h,\gamma)$

$$\Rightarrow (g_1,\beta_1)(h,\gamma)(p,\omega) = (g_2,\beta_2)(h,\gamma)(p,\omega)$$

$\Rightarrow (g_1, \beta_1) = (g_2, \beta_2)$ by (5)

Hence $(f_1, \alpha_1) : (X, \mu) \to (I, \xi)$ is an epimorphism.

Proof of the Theorem. From 3.2 and Lemma 3.3, it follows that $\mathcal{F}$ has epimorphic images.

Remark 3.4: We can prove that any two images are isomorphic fuzzy groups. Hence for practical purposes the image of $(f, \alpha) : (X, \mu) \to (Y, \eta)$ will be taken as $(f(X), \eta_1)$ where η_1 is the restriction of η.

Theorem 3.5: The category of fuzzy groups has inverse images.

Proof.

Let $P = \{ x \in X / f(x) \in u(Z) \}$

Then P is a subgroup of X. For if $x_1, x_2 \in P$ and $x_1 = u(z_1), x_2 = u(z_2)$ where $z_1, z_2 \in Z$, then

$$x_1 x_2^{-1} = u(z_1)(u(z_2))^{-1} = u(z_1)u(z_2^{-1})$$
$$= u(z_1 z_2^{-1}) \in u(Z)$$

so that $x_1, x_2^{-1} \in P$ and hence P is a subgroup of X.

Define $\epsilon : P \to [0, 1]$ as $\epsilon(x) = \mu(x)$, for all $x \in$ P, that is $\epsilon = \mu / P$. Then (P, ϵ) is a fuzzy subgroup of (X, μ).

Consider $(i_P, i_{\epsilon(P)}) : (P, \epsilon) \to (X, \mu)$. From the definition of ϵ, we see that for all $x \in P$, $\mu \, i_P(x) = \mu(x) = \epsilon(x) = i_{\epsilon(P)}\epsilon(x)$ so that $\mu \, i_P = i_{\epsilon(P)} \, \epsilon$.

Hence $(i_P, i_{\epsilon(P)}) : (P, \epsilon) \to (X, \mu)$ is a fuzzy group homomorphism.

Moreover since $i_{\epsilon(P)} : \epsilon(P) \to \mu(X)$ is injective.

$(i_P, i_{\epsilon(P)}) : (P, \epsilon) \to (X, \mu)$ is a fuzzy subgroup.

Next we define a fuzzy group homomorphism $(p_1, \beta_1) : (P, \epsilon) \to (Z, \theta)$ as follows.

Now $x \in P \Rightarrow f(x) = u(z)$ for some $z \in Z$ (by definition of P).

Moreover $u(z_1) = u(z_2) \Rightarrow z_1 = z_2$ since u is injective.

Thus $x \in P \Rightarrow$ there is a unique $z \in Z$ such that $f(x) = u(z)$.

Define $p_1(x) = z$ if $f(x) = u(z)$ (12)

Again $x \in P \Rightarrow f(x) = u(z), z \in Z$

$$\Rightarrow \eta f(x) = \eta u(z)$$
$$\Rightarrow \alpha\mu(x) = \eta u(z) \quad [\text{since } \alpha\mu = \eta f]$$
$$\Rightarrow \alpha\epsilon(x) = \eta u(z) \quad [\text{since } \epsilon = \mu / P]$$
$$\Rightarrow \alpha\epsilon(x) = \delta\theta(z) \quad [\text{since } \eta u = \delta\theta]$$

Also $\delta\theta(z_1) = \delta\theta(z_2) \Rightarrow \theta(z_1) = \theta(z_2)$ (since (u, δ) is a fuzzy subgroup $\Rightarrow \delta$ is injective)

Thus given $x \in P$, there is a unique $\theta(z) \in \theta(Z)$ (z need not be unique) such that $\alpha\epsilon(x) = \delta\theta(z)$.

Define $\beta_1 : \epsilon(P) \to \theta(Z)$ as $\beta_1\epsilon(x) = \theta(z)$ if $\alpha\epsilon(x) = \delta\theta(z)$ (13)

Claim 1. $(p_1, \beta_1) : (P, \epsilon) \to (Z, \theta)$ is a fuzzy group homomorphism.

Now for all $x \in P$, $\theta p_1(x) = \theta(z)$ where $f(x) = u(z)$ [by (12)]

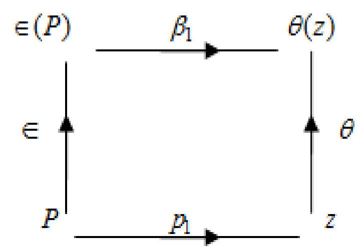

fig. 8

Also $\alpha\epsilon(x) = \delta\theta(z)$ so that $\beta_1\epsilon(x) = \theta(z)$ by (2)

Hence $\theta p_1(x) = \theta(z)$ where $f(x) = u(z)$ from (3).

Thus for all $x \in P$, $\beta_1\epsilon(x) = \theta P_1(x)$ so that $\beta_1\epsilon = \theta P_1$.

Hence the claim 1.

Claim 2. $(u, \delta)(p_1, \beta_1) = (f, \alpha)(i_P, i_{\epsilon(P)})$

Now for all $x \in P$, $up_1(x) = u(z)$, if $f(x) = u(z)$ by (1)

$$= f(x)$$
$$= f \, i_P(x)$$
$$\Rightarrow u \, p_1 = f \, i_P \quad \text{(A)}$$

Also for all $x \in P, \delta\beta_1\epsilon(x) = \delta\theta(z)$ where $\beta_1\epsilon(x) = \theta(z)$ if $\alpha\epsilon(x) = \delta\theta(z)$ by (2)

Hence $\delta\beta_1\epsilon(x) = \delta\theta(z)$

$$= \alpha\epsilon(x)$$
$$= \alpha \, i_{\epsilon(P)}\epsilon(x) \quad \text{for all } x \in P.$$

Therefore $\delta\beta_1 = \alpha \, i_{\epsilon(P)}$ (B)

From (A) and (B) we get claim (2)

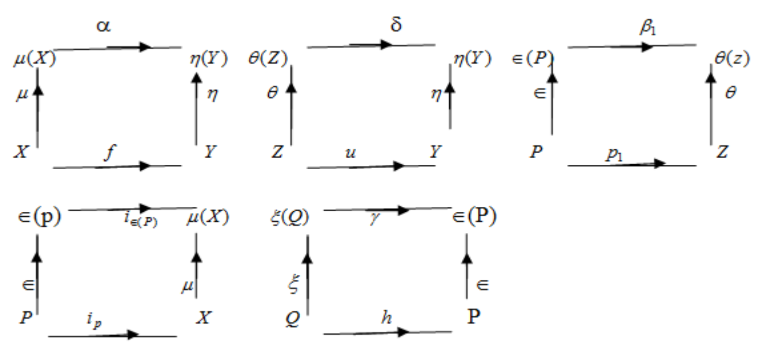

fig. 9

Suppose there exists $(q_1, \delta_1) : (Q, \xi) \to (Z, \theta)$ and $(q_2, \delta_2) : (Q, \xi) \to (X, \mu)$ such that

$(u, \delta)(q_1, \delta_1) = (f, \alpha)(q_2, \delta_2)$ (14)

We define $(h, \gamma) : (Q, \xi) \to (P, \epsilon)$ as follows

Let $t \in Q$. Then $q_2(t) \in X$ and $q_1(t) \in Z$ such that $f q_2(t) = u q_1(t) \in u(z)$.

Hence $q_2(t) \in P$ (by definition of P) and

$p_1 q_2(t) = q_1(t)$ by definition of p_1.

Define $h : Q \to P$ as $h(t) = q_2(t)$ (15)

Also define $\gamma : \xi(Q) \to \epsilon(P)$ by

$\gamma\xi(t) = \delta_2\xi(t)\ t \in Q$

Hence claim 2.

Claim 3 :

γ is well defined. (that is we have to prove that $\delta_2\xi(t)$ belongs to P)

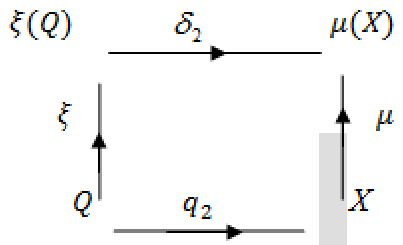

fig. 10

Now for all $t \in Q, \delta_2\xi(t) = \mu q_2(t)$

[since $\delta_2\xi = \mu q_2$] = $\epsilon q_2(t)$

[since $q_2(t) \in P$ and $\epsilon = \mu/_P$ and so $\delta_2\xi(t)$ belongs to $\epsilon(P)$ [since $q_2(t) \in P$]

Thus γ is well defined.

Moreover for all $t \in Q$

$\gamma\xi(t) = \delta_2\xi(t)$

$= \mu q_2(t)$ [since $(q_2, \delta_2) : (Q, \xi) \to (X, \mu)$ is a fuzzy group homomorphism]

$= \epsilon q_2(t)$ [since $q_2(t) \in P$ and $\epsilon = \mu/_P$]

$= \epsilon h(t)$ [by definition of h)

Hence $\gamma\xi = \epsilon h$ so that $(h, \gamma) : (Q, \xi) \to (P, \epsilon)$ is a fuzzy group homomorphism. Finally by definition of h and γ we have $i_P\, h = q_2$ and $i_{\epsilon(P)}\, \gamma = \delta_2$ so that

$(i_P,\ i_{\epsilon(P)})(h, \gamma) = (q_2, \delta_2)$ (16)

Claim 4 :

$(p_1, \beta_1)(h, \gamma) = (q_1, \delta_1)$

Now for all $t \in Q, p_1 h(t) = p_1 q_2($ (by definition of h)

$= q_1(t)$

and hence $p_1 h = q_1$ (17)

Also for all $t \in Q, \beta_1\gamma(\epsilon(t)) = \beta_1\delta_2(\epsilon(t))$ (18)

and $\delta\beta_1\delta_2(\epsilon(t)) = \alpha\, i_{\epsilon(P)}\delta_2(\epsilon(t))$

[since $\delta\beta_1 = \alpha\, i_{\epsilon(P)}$]

$= \alpha\delta_2(\epsilon(t)) = \delta\delta_1(\epsilon(t))$

[since δ is injective]

We conclude that $\beta_1\delta_2(\epsilon(t)) = \delta_1(\epsilon(t))$ (19)

From (7) and (8) we conclude that $\beta_1\gamma(\epsilon(t)) = \delta_1(\epsilon(t))$ (Since δ is injective)

In other words $\beta_1\gamma = \delta_1$ (20)

Thus the category of fuzzy groups has inverse images.

Note 3.6:

As in any category we can prove that any two images / inverse images are isomorphic.

Remark 3.7: Since any two inverse images can be proved to be isomorphic fuzzy groups, from the construction in proposition 3.1.23. it follows that the inverse image of a fuzzy subgroup $(i_z, i_{\mu'(z)}) : (Z, \eta') \to (Y, \eta)$ by $(f, \alpha) : (X, \mu) \to (Y, \eta)$ can be taken as $(f^{-1}(Z), \mu')$ where $f^{-1}(z) = \{x \in X \text{ given } f(x) \in Z\}$ (set theoretic inverse image) and μ' is the inclusion.

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